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WHITE PAPER

Arbitrage-Free Interest Rate Model in Negative Rate Regimes

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Financial institutions often employ multiple interest rate models for different purposes, resulting in inconsistent model outputs.

This paper seeks to address the current challenges related to valuing financial instruments by introducing an arbitrage-free interest rate model that is consistent with a negative rate regime, transparent in rate distribution, and parsimonious in specifying parameters.

The described model has a broad range of applications, including derivative trading, portfolio management, and risk management.

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INTRODUCTION

Kochkodin, Bloomberg News, September 2019, reported that "negative interest rates broke the Black Scholes model, Pillar of Modern Finance" [6].

Consistent with Kochkodin's report, this paper shows how and why the current low-interest - rate regime requires us to evaluate the current implementation of interest rate models. Financial institutions often employ multiple interest rate models for different purposes, resulting in inconsistent model outputs. The process is arcane despite the central importance of models in financial institutions and in light of tremendous progress made in financial and information technologies.

Introducing an Interest Rate Model Applicable also in a Low Interest Rate Regime

This paper seeks to address the current challenges related to valuing financial instruments by introducing an arbitrage-free interest rate model that is consistent with a negative rate regime, transparent in rate distribution, and parsimonious in specifying parameters. The described model has a broad range of applications, including derivative trading, portfolio management, and risk management.

Re-evaluating interest rate models today is important because most interest rate models were developed in a much higher interest rate regime; few models had anticipated rates falling to 1.5%. An inappropriate interest rate model will affect the pricing not just fixed income derivatives, but also equity derivatives, loans, and bonds. The interest rate model also affects credit risk management under the Current Expected Credit Loss (CECL). Indeed, this paper shows that the market has already priced in negative rates in some swaptions in the current low-interest-rate regime in the US. And financial engineers can extend the proposed model for applications in a broad range of interest rate regimes.

The paper model can be used as a standardized model in option pricing because the model is transparent, specified by only five parameters. The model meets the standard requirements for interest rate models:

- accept negative interest rates with the distribution of rates bounded;
- exhibit mean reversion of interest and accept the market observed term structure of volatilities;
- calculate the greeks at all nodes because the model remains "differentiable" in all rate regimes, whereby ad hoc constraints of rate floors would fail; and
- calculate American option types by backward substitution, without using numerical optimization for rational option exercise that is required by non-combining paths and which may incur errors in negative interest rate regimes.

MODELING FRAMEWORK

Ho-Lee (1984) shows that the interest rate model is specified by the movement of a one period bond that is described by the one-period forward price with up and down movements of 1/(π(i, n) +(1- π(i, n))δ(i, n) and δ(i, n)/(π(i, n) +(1- π(i, n))δ(i, n)), respectively for each state i and time n. Therefore, the volatilities represented by δ(i, n) depend on state and time, and not necessarily constant. Ho-Mudavanhu (2007) uses this approach to specify the term structure of local volatilities, depending on both state and time, as described by equation (1)

$$\sigma(i,n)=f(n).g(i,n) \quad (1)$$

where

$$f(n)=(a+bn)\ \exp(-cn)+d \quad (2)$$

and

$$g(i,n)=\min(r(i,n),R) \quad (3)$$

Ho-Mudavanhu further presents the pricing models using local volatilities specifications. In sum, Ho-Lee shows that any lattice arbitrage-free interest rate model can be formulated by the forward rates and the local volatilities σ(i, n) at each state i = 0… n and time step n. This paper will use the Ho-Lee framework to present the Model that would still be applicable in today's low interest rate regime, as lognormal and other models fail.

LOCAL VOLATILITIES MODELS

This paper first proposes a class of arbitrage-free models by expressing σ(i, n) as

$$\sigma(i\,,n) = f(n).\,n.\,g(i,n;p)\sqrt{\Delta t} \quad (4)$$

Local Volatilities Models remain arbitrage-free in negative interest rate regimes and consistent with historical rate movements

where Δt is the step size of the lattice. For example, the monthly step-size would be 1/12. $f(n)$ is the term structure of volatilities, independent of the states i. g(i, n; p) is a discrete frequency distribution for a given time n and the distribution parameters p, where p can be a vector of parameters.

For example, the frequency distribution can be uniform with probability 1/n or the probability distribution can be a binomial with a probability parameter p.

Following Ho-Mudavanhu, the term structure of volatilities can be represented by

$$f(n) = (a + bn)\exp(-cn) + d \quad (5)$$

Let $\delta^n{}_i$ be defined as

$$\delta_i^n = \exp(-2f(n).\,n\,.\,g(i,n;p)\sqrt{\Delta t}) \quad (6)$$

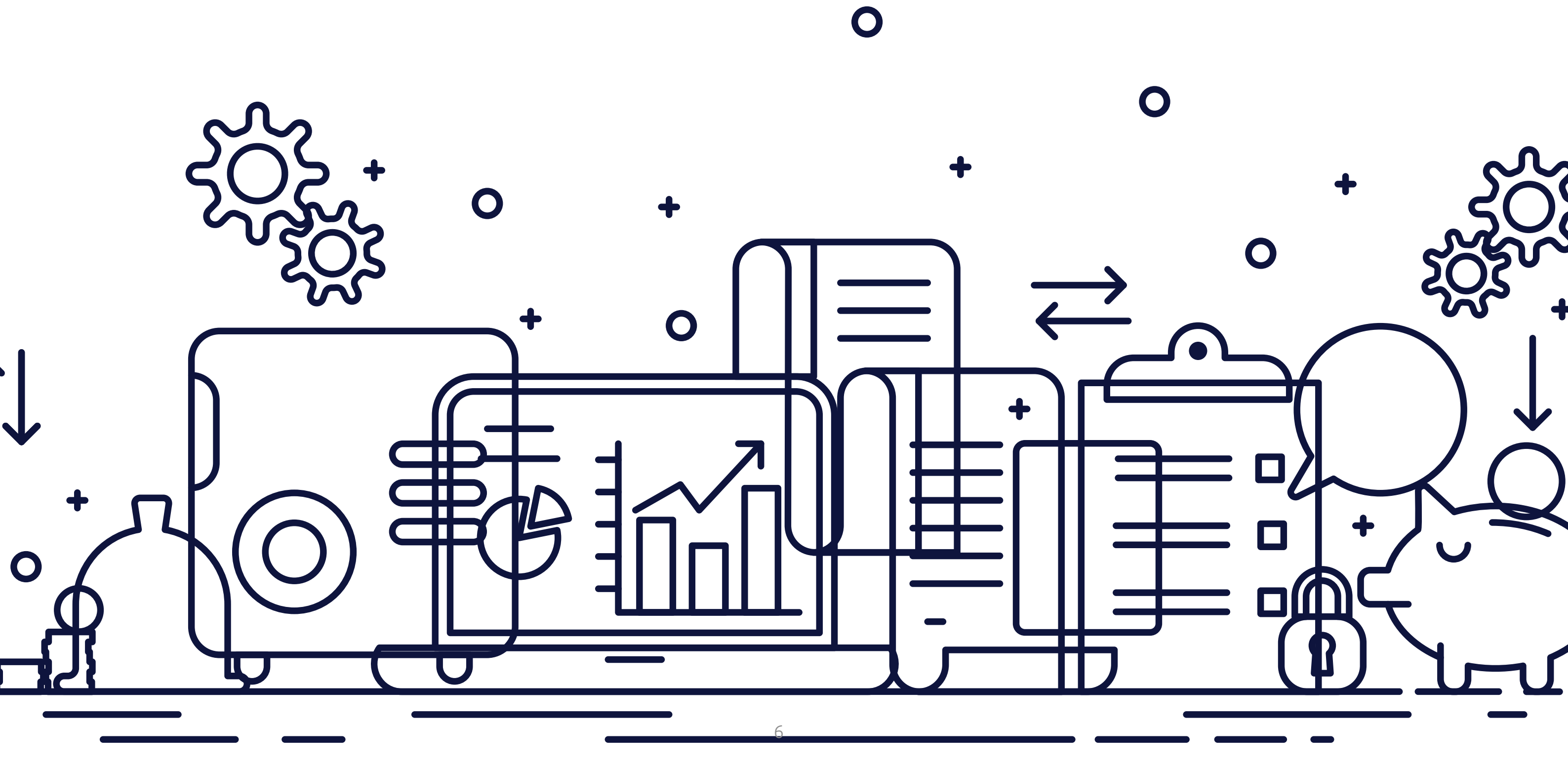
Ho-Mudavanhu shows that the arbitrage-free recombining lattice model requires that

$$\delta_i^n\,(T) = \delta_i^n\,\delta_i^{(n+1)}\,(T+1)((1+\delta_{(i+1)}^{(n+1)}\,(T-1))/(1+\delta_{(i+1)}^{(n+1)}\,(T-1)) \quad (7)$$

Then the one-period bond price can be constructed recursively using the equation below.

$$P_i^n = \frac{P(n+1)}{P(n)} \prod_{k=1}^{i} \frac{1+\delta_0^{k-1}(i-k)}{1+\delta_0^{k-1}(i-k+1)} \prod_{m=0}^{i-1} \delta_m^{n-1} \qquad (8)$$

Equations 5 to 8 complete the specification of the class of Local Volatilities Model. The first factor is the one-period forward price. The second factor represents the term premium as a result of uncertainties to ensure the rate movements are arbitrage free. The third term specifies the volatilities of interest rates. Explanations of these factors are provided in Ho-Lee (1984). Appendix A also provides the recursive algorithm in specifying equation (8) providing clarity in the exposition of the model.



RATE STRATIFICATION:

PROBABILITY FREQUENCY DISTRIBUTION OF RATES

Because interest rate models are central to many practices in the capital market, they should be transparent and validated. However, most models today remain black boxes to the users, providing no simple way to check the specifications or intuitive explanations of the model. Transparency is particularly important because the model efficacy may depend on the interest rate regime and the financial instruments being analyzed.

Local Volatilities Models determine the Rate Stratification that provides model transparency

This section uses equation (4) to show that the Local Volatilities Models can provide transparency of the calibration solution. Local Volatilites models can determine the risk neutral probability of the rates passing any segments on the lattice. Referring to equation (4) the local volatilities are given by

$$\sigma(i\,,n) = f(n).\, n.\, g(i, n; p)\sqrt{\Delta t}$$

Note that the term structure of volatilities f(n) is independent of the state i. Therefore f(n) is constant at the nth step for all i.

In particular, for a specified time n and g(i, n; p) a skewed binomial distribution, Appendix B provides the following equation (9):

$$\text{Prob}\ (x < r) = \frac{[r - r(\min,n)]}{[r(\max,n) - r(\min,n)]} \frac{\sum_{j=0}^{i*}\binom{n}{j}(0.5)^n}{\sum_{j=0}^{i*}\binom{n}{i}p^i(1-p)^{n-i}} \qquad (9)$$

Let the highest and lowest rate of the binomial lattice at time n specified by r(max, n) and r(min, n) respectively. Then the probability of rates below any specified rate r is dependent on (1) the ratio of r – r(min, n) to r(max, n) - r(min, n)).
Probability Stratification depicts the rate distribution of Equation (9). Figure 1 shows that the probability is related to the stratification

$$\frac{[r - r(\min,n)]}{[r(\max,n) - r(\min,n)]}$$

adjusted by the ratio of the cumulative binomial distribution of the normal to the skewed distribution. For a special case, when r = 0 then equation (9) measures the likelihood of negative rate occurring at time n as implied by swaptions or other interest rate option prices.

Referring to Figure 1, Equation (9) (next page) is important because the model provides transparency of the interest rate model. It provides the depiction of the probability distribution of rates at a particular future date, for example, the probability of the short rates lying between 6% and 7% in 10 years. For this reason, equation (9) provides a visual depiction of the changes in the interest rate model when the market yield curve changes in shape or the term structure of volatilities change with the market perceived uncertainties.

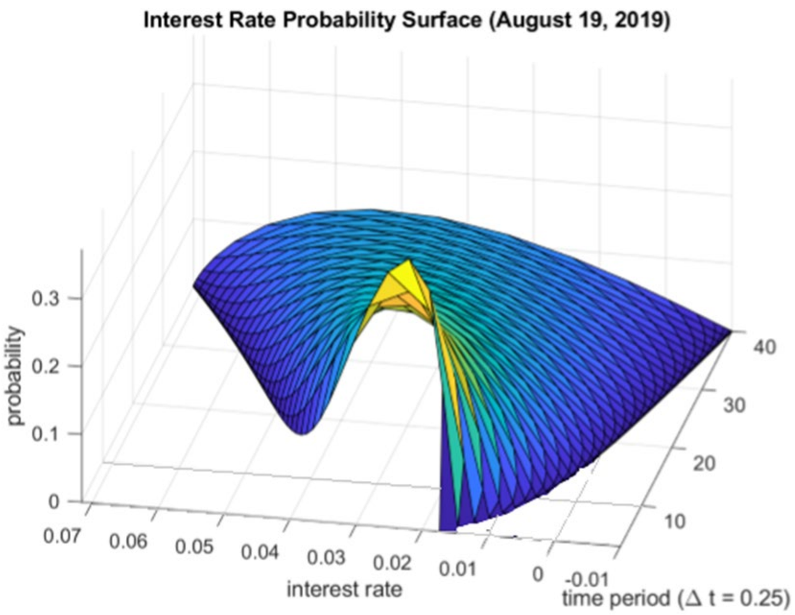


Figure 1. Model Interest Rate Probability Surface

Figure 1 depicts the probability frequency of rates reaching level r (x-axis) at time n, as the number of quarters. Dark blue denotes the low probability and green the high probability. The result shows that the rates evolve within a range of 1.5% and 4.0% with a wider range in 10 years.

Rate Stratification can be used to determine structured sampling for stress test and path dependent instrument pricing

Another application of equation (9) is to enable risk managers to selects interest rate scenarios for stress testing or income simulation. For quantitative modelling, the equation suggests stratified sampling technique for valuation. This is because equation (9) provides a simple metric to measure the likelihood of the selected scenario within the space of interest rate movements and to identify the comprehensiveness of the coverage of the selected scenarios of potential risks.

For example, some models run 240 Monte-Carlo simulations to value 1-4 family mortgages. If the model is using either lognormal or normal models, then the sparse sampling may mis-estimate the probabilities of the path space.

Further, the sample size must also depend on the yield curve regime. Therefore, evaluating the stratification of the rates is important. For example, Figure 1 shows that a larger number of sample paths may be required for longer term instruments, as the range of interest rate level is much wider as the time horizon increases.

CALIBRATED RATE SURFACES AND MODEL SPECIFICATION

In this section, we consider a particular arbitrage-free rate model where g(i; n, p) is the binomial probability with p. The model will be used to simulate results based on historical swaption prices to determine the rate surfaces. In particular, we let

$$g(i, n; p) = \binom{n}{i} p^{i} (1-p)^{n-i} \tag{10}$$

Figure 1 is a Rate Surface depicting the short rates of the lattice. The figure shows that the surface is not constrained by any ad hoc rate floors and is therefore consistent with arbitrage-free assumptions. At the same time, the rates do not become unreasonably negative nor unreasonably high.

1. RATE SURFACE

The stratification of the probabilities of the short rate reaching a particular range of rates is as explained by equation (9) and by Figure 1. Figure 1 and Figure 2 depict the stratification of rates under time n and state i of a binomial lattice. The two surfaces identify the interest rate region that is most important to the valuation of a particular financial instrument. The results show that any sparce sampling of interest rate paths in valuing path dependent instruments must consider the relevant interest rate region for valuation of the instruments. Figure 2 and Figure 3 provide that important information.

Rate Surface of the Local Volatilities Models can be visualized to ensure the reasonableness of the evolution of the interest rate movements

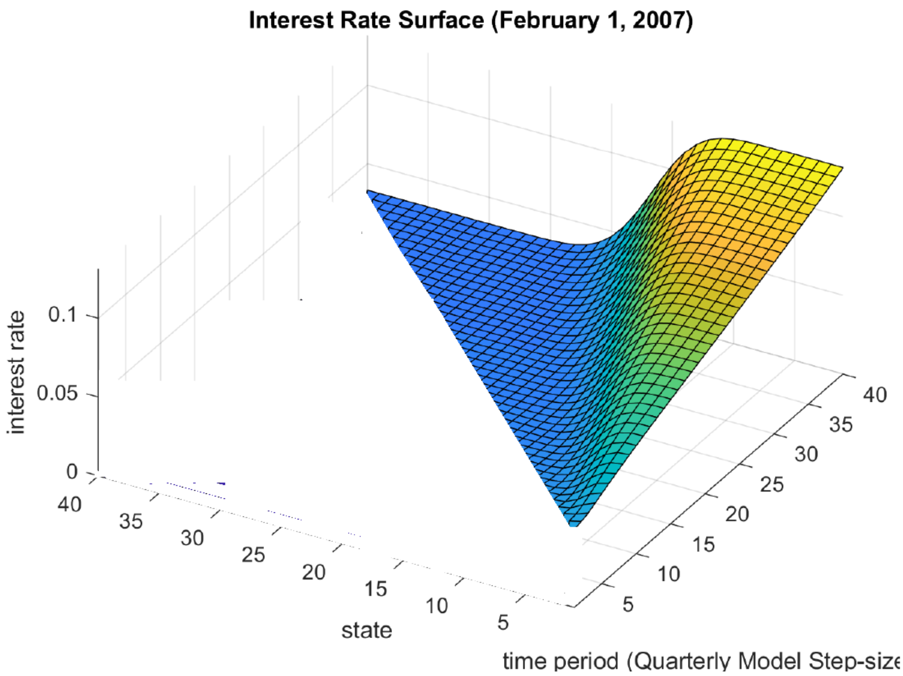


Figure 2. Rate Surface (February 1, 2007)

The rate surface depicts the rate level over a range of states i and time n, as the number of quarters. The results show that the evolution of the rate can exceed 10% and will remain positive in all states.

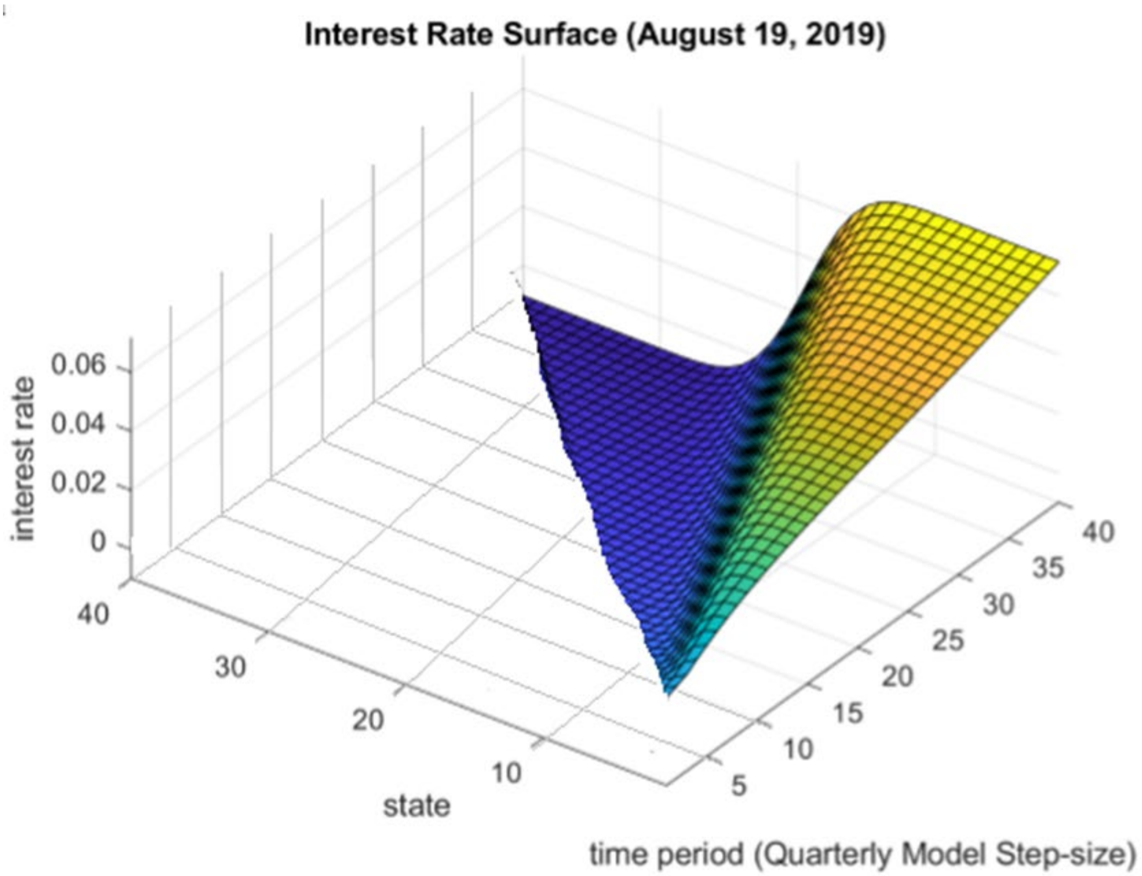


Figure 3. Rate Surface (August 19, 2019)

By way of contrast to the rate surface on February 01, 2007, the interest rate can be negative and the highest rate level is below 7% over ten years. The dark blue color represents the low rate level while yellow represents high rates.

The evolution of the short rate provides transparency of the model, ensuring that the rate movements exhibit reasonable behavior. Figure 2 seems to suggest that many interest rate paths lie in the negative rate region. But Equation (9) shows that the large number of paths in the negative rate region does not imply the market implied probability of reaching negative rates is high. The stratification of the negative rates is, in fact, small on this evaluation date.

2. MODEL SPECIFICATION

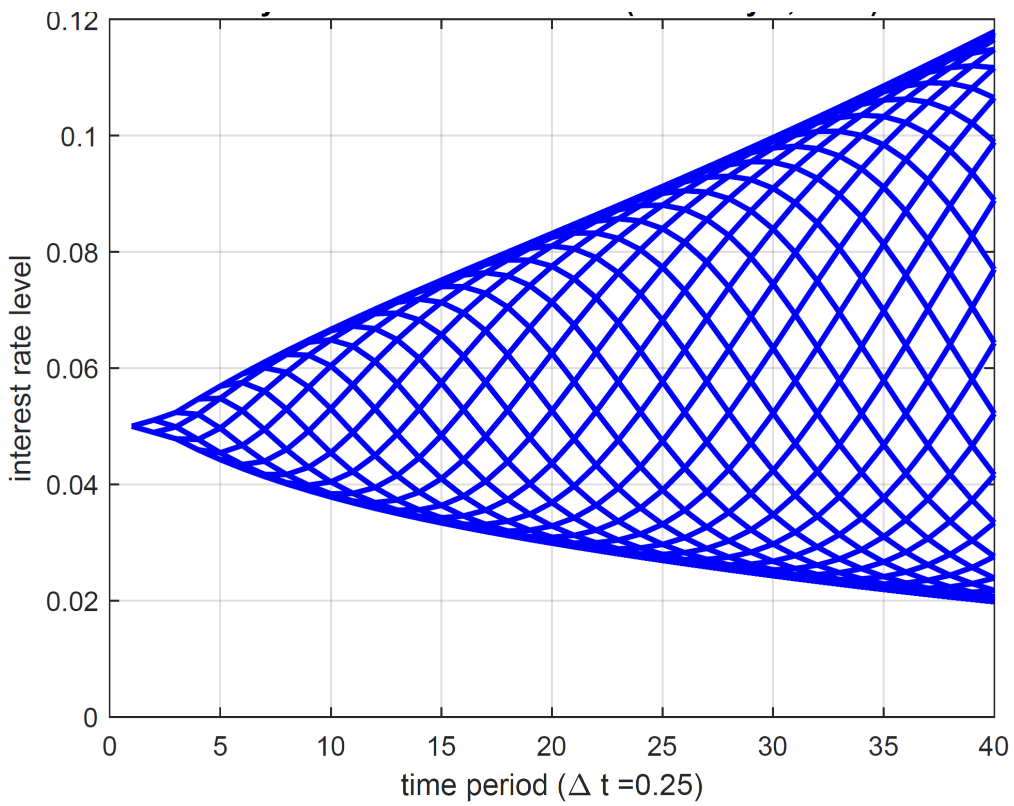
The Local Volatilities Models can be specified by only five parameters providing a standardized pricing methodology

Many current models involve complex specifications of interest rate models. For example, constraints on flooring interest rates, transiting from normal to lognormal models, and the use of lognormal models with adjustments for negative rates.

Model specification should be relatively simple while ensuring the model remains robust. The model is specified by five parameters: (i) a specifies the short term volatility; (ii) b specifies the short-term increase or decrease of volatility; (iii) c specifies the decay of the volatilities over time; (iv) d is the long term volatilities; (v) p is the skewness of the binomial distribution.

Historically, the change of short-term volatilities is subject to current market events, each of which may have a different lasting impact. For these reasons, parameters (a, b) are implied from the market pricing. Because of the difference between the perceived long term and short term uncertainties, parameter (c, d) captures the transition from the short term volatility to the long term volatility level. Finally, the volatilities have to depend on the rate levels, and the perceived dependence is captured by p.

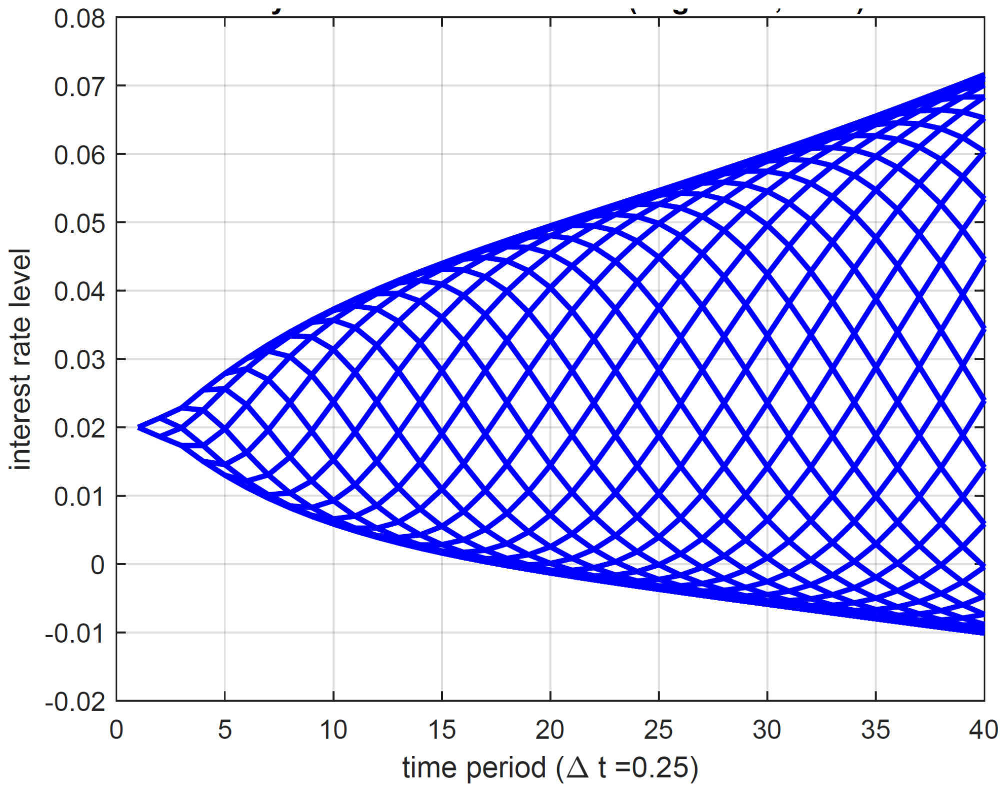
The figures below compare the rate surfaces at different historical times. Figure 2 and Figure 3 present the rate surfaces on February 1, 2007, and August 19, 2019. The calibrated rate surfaces show the impact of the upward sloping yield curve in 2007 versus the inverted yield curve of 2019 as the rates drift upward in the former and downwards in the latter. Also, note that the calibrated rate surface has negative interest rates.



Model Parameter	a	b	c	d	p
Calibrated values	0.0024	0.0000	0.0176	0.0025	0.4516

Figure 4. Rate Lattice (positive interest rates)

The results show that the long-term annualized volatility is 1.00% and the short-term volatility is 1.96% with a quarterly decay rate of 1.76%



Model Parameter	a	b	c	d	p
Calibrated values	0.0042	0.0000	0.0068	0.0020	0.4710

Figure 5. Rate Lattice (with negative interest rates)

The results show that the long-term annualized volatility is 0.80% and the short-term volatility is 2.48% with a quarterly decay rate of 0.68%. The calibrated high short-term volatility may be the perception that the Fed may lower the short-term interest rate, balancing the perception of a rise in inflation.

Figure 4 and 5 show 10-year interest rate lattices calibrated to 84 at-the-money swaptions for February 1, 2007 and August 19, 2019, respectively. The results show that in the low interest rate regime, as compared with the parameters of the high interest rate regime, the short-term and long term volatilities are lower. This is reasonable because volatilities should be positively related to the rate level. When volatilities are lower, the decay rate is also lower because the rate evolutions are more compressed. Finally, as expected, in the lower rate regime, the distribution is more skewed, as shown by the parameter p. The results seem reasonable.

The results have the following implications (i) in the low interest rate regime, lognormal models may not be providing correct option pricing because the market option prices have incorporated implied negative rates; (ii) state independent normal interest rate model would often have ad hoc minimum negative rate or zero rate, and the specification of the minimum rate can lead to inaccurate option pricing; (iii) trinomial lattice model requires adjustments to the probabilities assigned to node points, resulting in a more complicated calibration procedure; (iv) non-recombining interest rate models will require validation of the accuracy in estimating the rational option exercise rules for American type options in the negative rate regimes.

PRACTICAL IMPLICATIONS OF INTEREST RATE MODELS IN A LOW INTEREST RATE REGIME

This paper's results can address some of the practical issues in the current low interest regimes. We will consider the practical implications in the transactional markets, portfolio management, and financial reporting. To illustrate, we consider the implications of using "lognormal model" or "normal model." A lognormal model is defined as an interest rate model with volatility proportional to the rate level. Therefore, as rates fall, the volatility that the model assumes would become negligible. The normal model assumes the rate volatility is independent of the rate level and therefore the model rates can become significantly negative. To avoid the unlikely scenarios of extremely low rates, the model assumes the floor barrier. But there is no consensus on how low the floor rate should be. These models follow the traditional modelling approach of not assuming volatility is state dependent and that rates cannot become negative.

TRANSACTIONAL MARKET

In consumer loan lending, borrowers may not consider a 75 basis point rate floor valuable. But in the capital market floor derivatives are valuable, as the results above show the implied rates may even be negative. In this case, borrower customers tend not to require a lower loan rate for such floor, but the floor can be sold or marked to market with value.

The proposed model enables counterparties to transact using a meaningful price quoting standard particularly in a low interest rate regime.

Likewise, the capital market structured funding, such as FHLB Structured Put Advances, would have pricing discrepancy depending on the model used. Lognormal models would assign lower value to the floor than that of the normal model. Similarly, in the derivative market, the caps/floors or swaptions pricing would depend on the choice of models. But quoting option price based on the vol (volatility) is problematic because the quoted prices continue to diverge from the market implied volatilities. The current practice does not provide counterparties a meaningful price quoting standard. Currently, the practice has to quote the price based on OIS or LIBOR discounting. Current practice also requires disclosing normal or lognormal vol. This mechanism for transacting is inefficient.

PORTFOLIO MANAGEMENT AND FINANCIAL REPORTING

The proposed model provides a standardized model for many applications within a financial institution.

The capital market often uses Option Adjusted Spread (OAS) as a relative valuation tool. But OAS value depends on the underlying interest rate model. For floating rates, the OAS would be tighter in a normal model (in part depends on the floor set for such a model) than that of the lognormal. Likewise, for Mortgage-Backed Securities or 1-4 family 30-year fixed-rate mortgages, the prepayment speed would be affected. Many firms use multiple models for different purposes. As a result, fixed income portfolio management may not measure profitability consistently; the front desk profitability measures differ from those for financial reporting.

The proposed model is consistent with CECL accounting principles.

The interest rate model also affects the measure of credit. Current Expected Credit Loss (CECL) is a life-of-loan credit loss concept. As explained, the lognormal or normal models can affect the expected life of loans. For example, if a normal model assumes a floor of 75 basis points, an Adjustable Rate Mortgage would be assumed to be equivalent to a fixed-rate mortgage with an interest rate of 70 basis points, and in turn affect the life-of-loan measure of the Adjustable Rate Mortgage. The change in perceived life-of-loan will affect the CECL value for financial reporting. Likewise, the amortization of premiums will be affected by the choice of model. These practical issues should also affect financial reporting.

These are just some examples illustrating the importance of having a consistent interest rate model that is transparent, and that can be applicable over a broad range of interest rate regimes and applications.

CONCLUSIONS

This paper presents a state-time dependent local volatilities interest rate model based on Ho-Lee 1984. The model has the following attributes:

- *Only five parameters, a, b, c, d, and p that specify the model. Therefore, the model is transparent, which is an important attribute for an interest rate model, enabling counter-parties to establish a standard pricing model.*
- *The rate surface provides a simple validation of the model calibrated results*
- *Exhibit mean reversion process*
- *Allows for negative rates while satisfying arbitrage-free conditions*
- *The rates do not increase explosively as in a lognormal model, and the model remains arbitrage-free without flooring of interest rates. The model provides computational efficiency for American options without numerical approximation errors.*

The Model belongs to the class of Local Volatilities Models that can accept negative interest rates while consistent with empirical evidence of interest rate movements.

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Appendix A: Recursive Algorithm

For clarity of exposition, we present a recursive algorithm to generate the one period bond prices P_i^n, using equations (7) and (8). The recursive algorithm begins with $n = 0$ and $i = 0$ from equation (6). We can solve for δ_0^0 given that $\delta_0^0 = \frac{-\ln P_0^0(1)}{\Delta t}$, where $P_0^0(1) = P(1)$. Note that equation (7) is automatically satisfied given that $\delta_i^n(0) = 1$.

We now proceed with the recursive procedure. Let $n = 1$, equation (8) becomes,

$$P_i^1 = \begin{cases} \dfrac{P(2)\,(1+\delta_0^0(0))}{P(1)\,(1+\delta_0^0(1))} = \dfrac{2\,P(2)}{P(1)\,(1+\delta_0^0(1))}\,, & \text{for } i = 0 \\ P_0^1 \delta_0^0(1), & \text{for } i = 1 \end{cases}$$

Now, we substitute the value P_0^1 and P_1^1 to derive forward volatilities for period $n = 1$ and for $i = 0 \text{ or } 1$ using (6).

Now we repeat this procedure for period 2, $n = 2$. Using equation (7), we can now derive the forward volatilities for term $T = 2$ for $n = 0, 1$ and $0 \leq i \leq n$.

We now use equation (7) to determine the one period bond prices,

$$P_i^2 = \begin{cases} \dfrac{2\,P(3)\,(1+\delta_0^0)}{P(2)\,(1+\delta_0^0(2))\,(1+\delta_0^1)}, & \text{for } i = 0 \\ P_0^2\,\delta_0^1, & \text{for } i = 1 \\ P_0^2 \delta_0^1\,\delta_1^1, & \text{for } i = 2 \end{cases}$$

Using the one period bond prices and equation for δ, we can determine the one period forward volatilities δ_i^2, for $i = 0, 1, 2$.

This completes the specifications for $n = 2$ and we proceed to period 3, $n = 3$. Once again, we begin with equation (7) to determine the forward volatilities of terms greater than 1.

Then we apply (8) to determine the one period bond price for state 0.

$$P_0^3 = \frac{2\,P(4)\,(1+\delta_0^0(2))\,(1+\delta_0^1)}{P(3)\,(1+\delta_0^0(3))\,(1+\delta_0^1(2))(1+\delta_0^2)}$$

and

$$P_i^3 = P_0^3 \prod_{k=1}^{i} \delta_{k-1}^2$$

for $i = 1, 2, 3$.

From the one period bond prices, we determine one period forward volatilities δ_i^3, for $i = 0, 1, 2, 3$. This process continues for all maturities , time periods *n* and all states *i*.

Appendix B: Rate Stratification

This appendix derives the rate stratification based on a Local Volatilities model. Suppose the max rate and min rate on the lattice at n are r(max) and r(min), respectively. Then

$$\frac{[\mathrm{r(max,n)} - \mathrm{r(min,n)}]}{\mathrm{n}} = \mathrm{f(n)} \qquad \text{(B.1)}$$

For a given rate r(i, n), the spread between r(i, n) and r(min, n) is given by, according to equation (4). For clarity of exposition, we assume that $\sqrt{\Delta t} = 1$

$$\mathrm{r(i,n)} - \mathrm{r(min,n)} = \sum_{j=0}^{i} \mathrm{f(n)} * \mathrm{n} * \mathrm{g(j,n;p)} \qquad \text{(B.2)}$$

Substitute f(n) in equation (9) into equation (10), we get

$$[\mathrm{r(i,n)} - \mathrm{r(min,n)}] / [\mathrm{r(max)} - \mathrm{r(min)}] = \sum_{j=0}^{i} \mathrm{g(j,n;p)} \qquad \text{(B.3)}$$

Given a rate r, we can determine i* such that

$$\sum_{j=0}^{i*} \mathrm{g(j,n;p)} = [\mathrm{r} - \mathrm{r(min,n)}] / [\mathrm{r(max,n)} - \mathrm{r(min,n)}] \qquad \text{(B.4)}$$

For clarity of exposition, Equation B.4 abuses the notation. Equation B.4 cannot be exactly equal both sides of the equation. i* only seeks to determine the left-hand side closest while remaining lower than the right-hand side.

Since the lattice of movement assume upward and downward probability is 0.5, the probability of rates falling below r* is given by the cumulative binomial distribution,

$$\text{Prob (r < r*) = } \sum_{j=0}^{i*} \binom{n}{j} 0.5^{n} \qquad \text{(B.5)}$$

Equation (B.4) can be rewritten as

$$\sum_{j=0}^{i*} \binom{n}{j} 0.5^{n} = \frac{[\mathrm{r} - \mathrm{r(min,n)}]}{[\mathrm{r(max,n)} - \mathrm{r(min,n)}]} \frac{\sum_{j=0}^{i*} \binom{n}{j} 0.5^{n}}{\sum_{j=0}^{i*} \mathrm{g(j,n;p)}} \qquad \text{(B.6)}$$

Substituting Equation (B.6) into Equation (B.5), we have derived

$$\text{Prob (x < r) = } \frac{[\mathrm{r} - \mathrm{r(min,n)}]}{[\mathrm{r(max,n)} - \mathrm{r(min,n)}]} \frac{\sum_{j=0}^{i*} \binom{n}{j} 0.5^{n}}{\sum_{j=0}^{i*} \mathrm{g(j,n;p)}} \qquad \text{(B.7)}$$

When g(i, n; p) = $\binom{n}{i} p^{i} (1-p)^{n-i}$ then equation B.7 becomes

$$\text{Prob (x < r) = } \frac{[\mathrm{r} - \mathrm{r(min,n)}]}{[\mathrm{r(max,n)} - \mathrm{r(min,n)}]} \frac{\sum_{j=0}^{i*} \binom{n}{j} 0.5^{n}}{\sum_{j=0}^{i*} \binom{n}{i} p^{i} (1-p)^{n-i}} \qquad \text{(B.8)}$$

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